

# One-loop Shift in Noncommutative Chern-Simons Coupling

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## Abstract

In this paper we study the one-loop shift in the coupling constant in a non-commutative pure  $U(N)$  Chern-Simons gauge theory in three dimensions. The one-loop shift is shown to be a constant proportional to  $N$ , independent of noncommutativity parameters, and non-vanishing for  $U(1)$  theory. Possible physical and mathematical implications of this result are discussed.

## I. INTRODUCTION

Field theory (especially gauge field theory) on a noncommutative space (or spacetime) has attracted much interest recently [1–11]. That such theory arises from string/M(atr)ix theory [12,13] suggests that space or spacetime noncommutativity should be a general feature of quantum gravity for generic points deep inside the moduli space of M-theory. Moreover, being a natural deformation of usual quantum field theory, noncommutative field theory is of interests in its own right. A charge in the lowest Landau level in a strong magnetic field can be viewed as living in a noncommutative space, because the guiding-center coordinates of the charge are known not to commute. In this paper, we study noncommutative Chern-Simons (NCCS) field theory in 3 dimensions, which is a deformation of ordinary Chern-Simons (CS) theory and may have applications in planar condensed matter systems, especially in the quantum Hall systems. For simplicity, in this paper we mainly consider *pure* NCCS theory, with gauge group  $U(N)$  and with no matter fields coupled to it.

Three-dimensional noncommutative spacetime has coordinates satisfying

$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \quad \mu, \nu = 0, 1, 2, \quad (1)$$

where  $\theta^{\mu\nu}$  are antisymmetric and real parameters of dimension length squared. The action for a pure  $U(N)$  CS theory on this space reads

$$I_{CS} = -\frac{i\kappa}{4\pi} \int d^3x \varepsilon^{\mu\nu\lambda} \text{Tr}(A_\mu * \partial_\nu A_\lambda + \frac{2}{3} A_\mu * A_\nu * A_\lambda). \quad (2)$$

Here the dynamical field is the gauge potential  $A^\mu = A_\mu^a T^a$ ,  $T^a$  the generators of the gauge group  $G = U(N)$ , normalized to  $\text{Tr}(T^a T^b) = -\delta^{ab}/2$  with  $T^0 = i/\sqrt{2N}$  for the  $U(1)$  sector.  $\kappa$  is the CS coupling,  $\varepsilon^{\mu\nu\lambda}$  the totally antisymmetric tensor with  $\varepsilon^{012} = 1$ . In the action (2) we are using a representation, in which the coordinates  $x^\mu$  are the same as usual, but the product of any two functions of  $x^\mu$  is deformed to the Moyal star-product:

$$f * g(x) = e^{(i/2)\theta^{\mu\nu} \partial_\mu^x \partial_\nu^y} f(x)g(y)|_{y=x}. \quad (3)$$

The commutator in Eq. (1) is understood as the Moyal bracket with respect to the star product:

$$[x^\mu, x^\nu] \equiv x^\mu * x^\nu - x^\nu * x^\mu. \quad (4)$$

For applications to a system in the lowest Landau level, one considers only the spatial noncommutativity:  $\theta^{01} = \theta^{02} = 0$  and  $[x^1, x^2] = i\theta$ .

It is obvious that if  $\theta^{\mu\nu} = 0$ , the action (2) reduces to that of ordinary pure CS theory in 3 dimensions [14], which is known to be a topological quantum field theory [15], with the partition function and the correlation functions of Wilson loops being topological invariants, independent of spacetime metric. Diagrammatically the ordinary CS theory is renormalizable [16]. Many topological features can be probed in perturbation theory [17]. One interesting result is the one-loop quantum shift of the non-Abelian CS coupling [17]. However, ordinary pure Abelian CS theory has no such shift, though additional matter coupling does at the two-loop level [18–20]. In this paper we will show that there is a non-vanishing one-loop shift in noncommutative CS coupling even if the gauge group is  $U(1)$ . This shift turns out to be a constant proportional to the integer  $N$ , independent of the noncommutativity parameters  $\theta^{\mu\nu}$ , and identical to the one-loop shift in ordinary  $SU(N)$  CS theory when  $N \geq 2$ . Possible physical and mathematical implications of our results will be discussed.

## II. REGULARIZED FEYNMAN RULES

The action (2) is invariant under the following infinitesimal gauge transformations:

$$\delta A_\mu = D_\mu \lambda \equiv \partial \lambda + [A_\mu, \lambda]. \quad (5)$$

To do perturbation theory, we follow the standard procedure of path integral quantization to establish the Feynman rules. The full, regularized action after gauge fixing in Euclidean spacetime reads

$$I_{tot} = I_{CS} + I_{YM} + I_{gf} + I_{gh}. \quad (6)$$

Here we have added the noncommutative Yang-Mills (YM) term

$$I_{YM} = -\frac{1}{2e^2} \int d^3x \text{Tr}(F_{\mu\nu} * F^{\mu\nu}), \quad (7)$$

with the field strength  $F_{\mu\nu}$  defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]. \quad (8)$$

This gauge invariant term in the action provides a higher-derivative regularization for the CS theory, since the YM coupling  $e^2$  is of dimension of mass, which is used as a cut-off that is sent to infinity at the end of calculations. The third term in Eq. (6) is the gauge fixing term:

$$I_{gf} = -\frac{1}{\alpha e^2} \int d^3x \text{Tr}(\partial^\mu A_\mu)^2, \quad (9)$$

a linear, covariant gauge condition convenient for perturbation theory. In the following we are going to take the Landau gauge  $\alpha = 0$ , which was known to have computational advantages in the infrared in ordinary CS theory [16,20]. The last term  $I_{gh}$  is the ghost action corresponding to the above gauge fixing:

$$I_{gh} = \int d^3x \text{Tr}(\partial^\mu \bar{c} D_\mu c), \quad (10)$$

where  $c$  and  $\bar{c}$  are the ghost and anti-ghost respectively.

In ordinary Yang-Mills theory, to write down the Feynman rules, in addition to full action (6), one more thing one needs to know is the representation of the gauge group, since it determines the normalization of the group factors. In the following, we will mainly concentrate on  $U(1)$  theory, and we will come to  $U(N)$  case naturally after the explicit calculation for  $U(1)$  case. In noncommutative  $U(1)$  gauge theory, though the group factor is trivial, what is *nontrivial* is the noncommutativity of the kernel in Fourier transform. Suppose we have two kernels of Fourier transform  $\mathcal{F}_k = e^{ik \cdot x}$  and  $\mathcal{F}_p = e^{ip \cdot x}$ , by using Eq. (3), one can easily check the following commutator

$$[\mathcal{F}_k, \mathcal{F}_p] = -2i \sin\left(\frac{\theta^{ij}}{2} p_i k_j\right) \mathcal{F}_{k+p} = 2i \sin\left(\frac{\theta}{2} k \wedge p\right) \mathcal{F}_{k+p}. \quad (11)$$

where  $k \wedge p \equiv k_\mu \theta^{\mu\nu} p_\nu$ , and we have used Eq. (1). After making Fourier transform of the action, one can immediately find out that this commutator plays exactly the same role as that of Lie commutators of the gauge group in ordinary non-Abelian gauge theory. Therefore, we can establish the Feynman rules by following the same procedure as that of ordinary Yang-Mills theory, with the group structure constants,  $f^{abc}$ , being replaced by a momentum-dependent factor [2,3], namely,

$$f^{abc} \rightarrow f^{k,p,k+p} = \sqrt{2}i \sin\left(\frac{\theta}{2} k \wedge p\right), \quad (12)$$

where  $\sqrt{2}$  is due to the normalization of  $T^0$ . With the help of this correspondence, we establish the following Feynman rules for  $U(1)$  NCCS theory:

(i) The gluon propagator:

$$\Delta_{\mu\nu}(p) = \frac{4\pi}{\kappa} \frac{m}{p^2(p^2 + m^2)} (m\varepsilon_{\mu\nu\rho} p^\rho + \delta_{\mu\nu} p^2 - p_\mu p_\nu). \quad (13)$$

where  $m = e^2 \kappa / 4\pi$ . At the end of computations, we remove the cut-off by taking  $e^2 \rightarrow \infty$  or  $m \rightarrow \infty$ .

(ii) The ghost propagator:

$$\frac{1}{p^2}. \quad (14)$$

(iii) The ghost-ghost-gluon vertex:

$$- \sqrt{2} q_\nu \sin\left[\frac{\theta}{2} q \wedge p\right]. \quad (15)$$

(iv) The three gluon vertex:

$$\frac{\kappa}{4\pi} \frac{\sqrt{2}}{m} \sin\left[\frac{\theta}{2} p \wedge q\right] [m\varepsilon_{\mu\nu\rho} - (r - q)_\mu \delta_{\nu\rho} - (q - p)_\rho \delta_{\mu\nu} - (p - r)_\nu \delta_{\rho\mu}]. \quad (16)$$

(v) The four-gluon vertex:

$$\begin{aligned} \frac{\kappa}{4\pi} \frac{1}{m} [ & f^{p,q,t} f^{r,s,t} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) + f^{r,q,t} f^{p,s,t} (\delta_{\mu\rho} \delta_{\nu\sigma} - \delta_{\mu\nu} \delta_{\sigma\rho}) \\ & + f^{s,q,t} f^{r,p,t} (\delta_{\mu\nu} \delta_{\rho\sigma} - \delta_{\mu\sigma} \delta_{\nu\rho}) ]. \end{aligned} \quad (17)$$

where  $p, q, r, s, t$  are incoming momenta and  $f^{x,y,z}$  is given by Eq. (12).

### III. WARD-SLAVNOV-TAYLOR IDENTITIES

In ordinary gauge theories, Ward-Slavnov-Taylor (WST) identities play a very important role in renormalized perturbation theory. For renormalizable gauge theories, they are essentially manifestation of gauge invariance for the regularized and renormalized action (with counter terms included). Conversely, checking WST identities is essentially checking renormalizability and gauge invariance of the renormalized gauge theory. The same is true for noncommutative gauge theories. In the following we are going to check part of the Ward identities to assure gauge invariance, and to use part of them to simplify the calculations.

Renormalizability of the theory requires that the full inverse  $A$ -propagator and the full  $AAA$ -vertex are of the following form as the external momenta tend to zero

$$\Delta_{\mu\nu}^{-1}(k) \rightarrow \frac{\kappa}{4\pi} Z_A \epsilon_{\mu\nu\lambda} k_\lambda + Z'_A (k^2 \delta_{\mu\nu} - k_\mu k_\nu), \quad (k \rightarrow 0), \quad (18)$$

$$\Gamma_{\mu\nu\lambda}(p, q, r) \rightarrow Z_g \epsilon_{\mu\nu\lambda}, \quad (p, q, r \rightarrow 0). \quad (19)$$

These equations defines the relevant renormalization constants  $Z_A$ ,  $Z'_A$  and  $Z_g$ . In next section, we will confirm the validity of these equations for one-loop two-point and three-point functions, to verify renormalizability of the theory at the one-loop level. Similarly one can defined  $Z_{gh}$  and  $\tilde{Z}_g$ , the renormalization constants<sup>1</sup> for the ghost wave function and the  $\bar{c}Ac$ -vertex respectively, through the full ghost propagator and the full  $\bar{c}Ac$ -vertex

$$\tilde{\Delta}(p) \rightarrow \frac{1}{Z_{gh} p^2}, \quad (p \rightarrow 0), \quad (20)$$

$$i\Gamma_\mu(p, q, r) \rightarrow i\tilde{Z}_g p_\lambda, \quad (p, q, r \rightarrow 0). \quad (21)$$

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<sup>1</sup> Here we would like to remind that all the renormalization constants we defined here are consistent with the conventions used in the refs. [16,17,19,20], while being the inverse of the standard ones used in many textbooks on quantum field theory.

In the next section we will see that at least at one loop, these renormalization constants are in fact finite, i.e. they are independent of the cut-off.

Assuming renormalizability and introducing the renormalized fields and renormalization constants, one can write the renormalized action as

$$I_{ren} = I'_{CS} + I'_{gf} + I'_{gh} \quad (22)$$

with

$$I'_{CS} = -\frac{i\kappa_r}{4\pi} \int d^3x \varepsilon^{\mu\nu\lambda} \text{Tr}[Z_A^{-1} A_\mu^{(r)} \partial_\nu A_\lambda^{(r)} + \frac{2}{3} Z_g^{-1} A_\mu^{(r)} * A_\nu^{(r)} * A_\lambda^{(r)}], \quad (23)$$

$$I'_{gf} = -\frac{1}{\alpha_r} \int d^3x \text{Tr}(\partial^\mu A_\mu^{(r)})^2, \quad (24)$$

$$I'_{gh} = \int d^3x \text{Tr}(Z_{gh}^{-1} \partial^\mu \bar{c}^{(r)} \partial_\mu c^{(r)} + \tilde{Z}_g^{-1} \partial^\mu \bar{c}^{(r)} [A_\mu^{(r)}, c^{(r)}]). \quad (25)$$

The terms (23), (24) and (25) in the renormalized action (22) should be equal to, respectively, the corresponding terms (2), (9) and (10) in the original action (6), if they are expressed in terms of the bare fields through <sup>2</sup>

$$A_\mu = \frac{Z_A}{Z_g} A_\mu^{(r)}, \quad c = Z_{gh}^{-1/2} c^{(r)}, \quad \bar{c} = Z_{gh}^{-1/2} \bar{c}^{(r)}. \quad (26)$$

This requires that the renormalized CS coupling be related to the bare one by

$$\kappa_r = \frac{Z_A^3}{Z_g^2} \kappa, \quad (27)$$

and that the following WST identity be true for the renormalization constants:

$$\frac{Z_A}{Z_{gh}} = \frac{Z_g}{\tilde{Z}_g}. \quad (28)$$

As easy to check, the WST identities guarantee that the renormalized actions (23) and (25) are gauge invariant.

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<sup>2</sup>The relation between  $A_\mu$  and  $A_\mu^{(r)}$ , though looks unusual, is appropriate for the CS theory. See e.g. Refs. [16,17,20].

#### IV. ONE-LOOP RENORMALIZATION IN $U(1)$ THEORY

In this section, we first study at the one-loop level the renormalization of  $U(1)$  NCCS theory, in particular the shift in the Chern-Simons coupling  $\kappa$ .

Let us start with the ghost self-energy. It contains a planar diagram contribution

$$\tilde{\Pi}_p^{(1)} = \frac{4\pi}{\kappa} \frac{m}{p^2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2 p^2 - (k \cdot p)^2}{k^2(k^2 + m^2)(k + p)^2}, \quad (29)$$

and a non-planar diagram contribution

$$\tilde{\Pi}_{np}^{(1)} = \frac{4\pi}{\kappa} \frac{m}{p^2} \int \frac{d^3k}{(2\pi)^3} \frac{k^2 p^2 - (k \cdot p)^2}{k^2(k^2 + m^2)(k + p)^2} e^{i\theta k \wedge p}. \quad (30)$$

The integral (29) is finite, with the leading term proportional to  $1/|m|$ . Therefore, in the large  $|m|$  limit, we get the contribution to the ghost self-energy as

$$\tilde{\Pi}_p^{(1)} = -\frac{2}{3\kappa} \text{sgn}(\kappa). \quad (31)$$

On the other hand, the non-planar diagram contribution is evaluated to be

$$\tilde{\Pi}_{np}^{(1)} = -\frac{2}{3\kappa} \text{sgn}(\kappa) f(p\theta|m|), \quad (32)$$

where the function  $f(x)$  is defined by

$$f(x) = \frac{1}{x} \int_0^x dy (3y^{1/2} - y^{3/2}) K_{1/2}(y) = \sqrt{2\pi} \frac{(1+x)e^{-x} - 1}{2x}. \quad (33)$$

Since  $f(x) \rightarrow 0$  as  $x \rightarrow \infty$ , the non-planar diagram does not contribute to the ghost self-energy in the limit  $|m| \rightarrow \infty$ . Therefore, the ghost self-energy correction at the one loop level is finite and independent of external momentum  $p$ . Correspondingly, we get the one-loop ghost wave function renormalization constant

$$Z_{gh}^{(1)} = 1 - \frac{2}{3\kappa} \text{sgn}(\kappa). \quad (34)$$

To calculate the gluon self energy  $\Pi_{\mu\nu}^{(1)}(p)$ , we decompose it into the following structure

$$\Pi_{\mu\nu}^{(1)}(p) = \frac{1}{m} \Pi_e^{(1)}(\delta_{\mu\nu} p^2 - p_\mu p_\nu) + \frac{\kappa}{4\pi} \Pi_o^{(1)}(p) \varepsilon_{\mu\nu\lambda} p^\lambda \quad (35)$$



Since only the gluon loop diagram has an odd number of  $\varepsilon$  tensors,  $\Pi_o^{(1)}(p)$  receives a nonzero contribution only from the gluon loop diagram Fig. 2a. In contrast,  $\Pi_e^{(1)}$  picks up contributions from the tadpole diagram (Fig. 2c), the ghost loop diagram Fig. 2b, as well as from the gluon diagram Fig. 2a with an even number of  $\varepsilon$  tensors. Contracting  $\Pi_{\mu\nu}^{(1)}$  with  $\kappa/4\pi(\varepsilon_{\mu\nu\lambda}p^\lambda/2p^2)$  and  $m\delta_{\mu\nu}/2p^2$ , we obtain  $\Pi_o^{(1)}(p)$  and  $\Pi_e^{(1)}(p)$ :

$$\Pi_o^{(1)} = \frac{4\pi}{\kappa} \frac{2m}{p^2} \int \frac{d^3k}{(2\pi)^3} [\sin^2(\theta/2k \wedge p)] \frac{[k^2p^2 - (k \cdot p)^2][5k^2 + 5(k \cdot p) + 4p^2 + 2m^2]}{k^2(k+p)^2(k^2+m^2)[(k+p)^2+m^2]}, \quad (36)$$

and

$$\Pi_e^{(1)} = -\frac{m}{2p^2} \left[ \int \frac{d^3k}{(2\pi)^3} [\sin^2(\theta/2k \wedge p)] \frac{N_e(p, k)}{k^2(k+p)^2(k^2+m^2)[(k+p)^2+m^2]} + \frac{5m}{3\pi} \right]. \quad (37)$$

where

$$\begin{aligned} N_e(p, k) = & 6k^6 + 18k^4(k \cdot p) + 20k^4p^2 + 22k^2(k \cdot p)^2p^2 - 12(k \cdot p)^3 \\ & + 9k^2p^4 - 7(k \cdot p)^2p^2 + m^2[2k^4 + 4k^2(k \cdot p) + k^2p^2 + (k \cdot p)^2]. \end{aligned} \quad (38)$$

At this point we would like to comment that the structures shown in the above results are similar to those in ordinary non-Abelian Chern-Simons gauge theory in 2+1 dimensions, although here we are dealing with the  $U(1)$  case. Still they are different in the following two aspects. The first is that we have non-planar diagram contributions due to the oscillating factor  $4\sin^2(\theta/2k \wedge p)$ . The second is that the value of the tadpole contribution changes (see the second term in (37)), the reason being that one of the terms in the four-gluon vertex vanishes due to the fact  $\sin(\theta/2p \wedge p) = 0$  by using the Feynman rule (17).

The integral in Eq. (36) is finite. To calculate it, again we separate it into planar and non-planar contributions. The calculation of the planar contribution is standard. Taking  $|m| \rightarrow \infty$ , we obtain

$$\Pi_{o,p}^{(1)} = \frac{7}{3\kappa} \text{sgn}(\kappa). \quad (39)$$

By using Feynman parameterization, we can rewrite the corresponding non-planar contribution as follows:

$$\Pi_{o,np}^{(1)} = -2p^2 \int_0^1 dx \int_0^x dy \int_0^y dz \{5I_2(\nu) + [5p^2(1+y-x-z)(y-x-z) + 4p^2 + 2m^2]I_1(\nu)\}, \quad (40)$$

where the argument  $\nu$  in the functions  $I_1$  and  $I_2$  is defined as

$$\nu = \sqrt{p^2(1+y-x-z) + m^2(1-y)}. \quad (41)$$

The functions  $I_1$  and  $I_2$  are defined as

$$I_1 = \frac{\pi^{3/2}}{2} \left(\frac{\theta p}{2}\right)^{3/2} [\nu^{-3/2} K_{3/2}(\nu \theta p) - \frac{\theta p}{3} \nu^{-1/2} K_{1/2}(\nu \theta p)], \quad (42)$$

and

$$I_2 = \frac{\pi^{3/2}}{12} \left(\frac{\theta p}{2}\right)^{1/2} [15\nu^{-1/2} K_{1/2}(\nu \theta p) - 20\left(\frac{\theta p}{2}\right)^{3/2} \nu^{1/2} K_{1/2}(\nu \theta p) + 4\left(\frac{\theta p}{2}\right)^{5/2} \nu^{3/2} K_{3/2}(\nu \theta p)], \quad (43)$$

where  $K_\beta(x)$  is the modified Bessel function. It has an exponentially decay profile. In the limit  $|m| \rightarrow \infty$ , we see that the integrand in Eq. (40) vanishes, therefore the non-planar diagram does not contribute to  $\Pi_o^{(1)}$ . Namely,

$$\Pi_{o,np}^{(1)} = 0. \quad (44)$$

Thus, we get

$$\Pi_o^{(1)} = \frac{7}{3\kappa} \text{sgn}(\kappa). \quad (45)$$

Similar analysis can be applied to the integral (37). It turns out that the integral is finite as  $|m| \rightarrow \infty$ . Therefore, the photon wave function renormalization constant is

$$Z_A^{(1)} = 1 + \Pi_o^{(1)} = 1 + \frac{7}{3\kappa} \text{sgn}(\kappa). \quad (46)$$

Furthermore, we study the one loop corrections to the vertex  $\bar{c}Ac$ , we show that the one loop correction vanishes as  $|m| \rightarrow \infty$ . The reason is that the one- $\varepsilon$  term of Fig. 2(i) cancels against the three- $\varepsilon$  term of Fig. 2(h); the one- $\varepsilon$  term of Fig. 2(h) goes to zero; the non- $\varepsilon$

terms in the two diagrams cancel each other as well. Finally, we get the renormalization for the  $\bar{c}Ac$  vertex

$$\tilde{Z}_g^{(1)} = 1. \quad (47)$$

After we extract  $Z_{gh}^{(1)}$ ,  $Z_A^{(1)}$ , and  $\tilde{Z}_g^{(1)}$ , we can employ the WST identity (28) established in the previous section to get the three-gluon vertex renormalization constant:

$$Z_g^{(1)} = \frac{Z_A^{(1)}}{Z_{gh}^{(1)}} \tilde{Z}_g^{(1)} = 1 + \frac{3}{\kappa} \text{sgn}(\kappa), \quad (48)$$

where we have worked up to the first order in  $1/\kappa$ , consistent with one-loop perturbative theory.

Now we have shown that all renormalization constants at the one loop level are finite, so the one-loop beta function vanishes, as in ordinary CS theory. Substituting the renormalization constants  $Z_A^{(1)}$  and  $Z_g^{(1)}$  in the definition of the renormalized CS coupling  $\kappa_r^{(1)}$ , we have

$$\kappa_r^{(1)} = \kappa + \text{sgn}(\kappa). \quad (49)$$

This is the main result of the present paper. The second term is the desired one-loop shift in the  $U(1)$  Chern-Simons coupling. Note that the shift is just the unity in our normalization for the coupling. Note that it is independent both of the noncommutativity parameters  $\theta^{\mu\nu}$  and of the value of the bare coupling  $\kappa$  except for its sign. Also recall that for ordinary  $U(1)$  CS theory, the one-loop shift vanishes in the same  $F^2$  regularization.

## V. GENERALIZATION TO $U(N)$

In this section, we generalize the expression (49) we obtained in last section for the one-loop shift of the CS coupling from  $U(1)$  to  $U(N)$ .

It is known that for a noncommutative gauge theory, the gauge group is restricted to be only  $U(N)$ ; even  $SU(N)$  is not allowed, because the closure of the Moyal commutator is

violated [21]. Another way to see this is that the three-gluon coupling in the action (2) mixes the  $U(1)$  gluon with the  $SU(N)$  gluons. Thus, unlike ordinary gauge theory which allows the  $U(1)$  and  $SU(N)$  sectors to have independent coupling constants, in noncommutative theory  $U(N)$  gauge invariance enforces the  $U(1)$  and the  $SU(N)$  gluons to share the same coupling constant.

For  $U(N)$  noncommutative Yang-Mills (NCYM) theory, it is this distinct feature that makes the coupling in the  $U(1)$  sector runs in the same way as that in the  $SU(N)$  sector, as verified in a recent explicit calculation [11]. Before this calculation was done, a beautiful proof without doing any new calculation had been given in ref. [5] for the statement that the one-loop beta-function of  $U(N)$  NCYM can be simply read off from the known value of the ordinary  $SU(N)$  Yang-Mills theory. This is because in NCYM the nonplanar one-loop  $U(N)$  diagram contributes only to the  $U(1)$  part of the theory. In the following we will apply the same trick to NCCS, and derive the one-loop shift in the CS coupling in the  $U(N)$  theory without doing any new calculations.

Following ref. [5], let us consider the quadratic one-loop 1PI effective action of the ordinary  $U(N)$  CS theory, in which the  $U(1)$  and  $SU(N)$  sectors share the same coupling constant, in the  $F^2$  regularization. From the results in refs. [17,20] one infers that after removing the cut-off,

$$\Gamma_2^{(1)}(\theta = 0) = -\frac{i}{4\pi} \int d^3x \varepsilon^{\mu\nu\lambda} [(\kappa + N \text{sgn}(\kappa)) \text{Tr}(A_\mu \partial_\nu A_\lambda) - \text{sgn}(\kappa) (\text{Tr} A_\mu) \partial_\nu (\text{Tr} A_\lambda)]. \quad (50)$$

The coefficient  $\kappa + N \text{sgn}(\kappa)$  in the first term is read off from the known one-loop shift in ordinary  $SU(N)$  CS theory [17,20], while the existence of the second term is due to the necessity for cancelling the  $U(1)$  part in the first term, since we know there is no one-loop shift in the  $U(1)$  coupling constant. Also it is easy to check that the second term is the only nonplanar contribution to  $\Gamma_2^{(1)}$ , coming from the nonplanar part of diagrams like Fig. 2 (a)-(c).

Now let us turn on nonzero  $\theta_{\mu\nu}$ . The planar contributions to  $\Gamma_2^{(1)}$  are known to be the same as in ordinary theory [4,5], while the nonplanar diagrams are suppressed to zero by an

extra rapidly oscillating phase factor. Our result (44) in last section verifies the latter by explicit computation. Thus, with the second term put to zero, one reads from eq. (50) that in  $U(N)$  NCCS,

$$\Gamma_2^{(1)}(\theta \neq 0) = -\frac{i}{4\pi} \int d^3x \varepsilon^{\mu\nu\lambda} [\kappa + N \text{sgn}(\kappa)] \text{Tr}(A_\mu \partial_\nu A_\lambda). \quad (51)$$

Therefore, without any new calculation, we infer that the one-loop shift in the coupling for  $U(N)$  NCCS is

$$\kappa_r = \kappa + N \text{sgn}(\kappa). \quad (52)$$

For the  $U(1)$  case, we plug  $N = 1$  in Eq. (52), reproducing exactly Eq. (49) that we have obtained by explicit calculation in last section.

In summary, the one-loop shift of the CS coupling in  $U(N)$  NCCS is the same as that in ordinary  $SU(N)$  CS theory for  $N \geq 2$ , while for the  $U(1)$  case it gives a non-vanishing value in contrast to ordinary CS theory.

## VI. CONCLUSIONS AND DISCUSSIONS

The induced CS coupling by fermionic fields in noncommutative quantum electrodynamics in 3 dimensions has been studied in ref. [9]. In this paper we have considered instead a pure CS theory without matter, and have studied the one-loop quantum correction to the coupling constant due to self-interactions of the gauge bosons that arise from spacetime coordinate noncommutativity.

First of all, the renormalization constants we have calculated at the one loop level are finite, showing that the beta function at this level vanishes. Moreover all renormalization constants, including the one-loop shift in the CS coupling are shown to be independent of the noncommutativity parameters. This is a bit surprising, since the spacetime noncommutativity parameters appear in the Lagrangian of pure CS theory explicitly. This adds explicit evidence to a theorem proved in Ref. [10] that one-loop results in noncommutative

CS theory are all independent of spacetime noncommutativity. It would be interesting to see whether the same independence remains true at higher orders. Our conjectured answer is "yes". In other words, we conjecture that *noncommutative CS theory is a "deformed" topological field theory*, in the sense that the partition function and correlation functions are topological invariants, independent of both metric and noncommutativity parameters.

Furthermore, we have shown that the one-loop shift in the coupling constant does not vanish in a pure  $U(1)$  NCCS theory. This arises as a consequence of spacetime coordinate noncommutativity, since in ordinary  $U(1)$  CS theory there is no quantum shift in the coupling constant at all. We notice two features of our result (49): 1) it is independent of the bare coupling  $\kappa$ , except for its sign; 2) it is the simplest integer, the unity, independent of the noncommutativity parameters. Therefore this result is not smooth in the limit  $\theta_{\mu\nu} \rightarrow 0$ .

Finally, we have shown that the one-loop shift of the  $U(N)$  NCCS coupling is the integer  $N$  that characterizes the gauge group  $U(N)$ , exactly the same as that in ordinary  $SU(N)$  CS theory for  $N \geq 2$ . In ordinary CS theory the quantization of the shift was interpreted [17,20] as being consistent with the topological quantization of the non-abelian CS coupling. The latter is known [14] to result from the topological fact that a large gauge transformation changes the CS action by a value proportional to the integer winding number of the gauge transformation, viewed as a map from the (compactified) spacetime to the gauge group. Whether the topological quantization of the CS coupling remains true in the noncommutative theory is not clear at all at this moment. However, our result (52) shows that the one-loop shift is still quantized in the noncommutative case. If one reverses the logic in the above reasoning for the ordinary CS theory, this result seems to indicate that possibly in noncommutative geometry there should be a counterpart of the concept of the usual winding number that remains integer-valued, and that the NCCS coupling should satisfy a similar topological quantization.

Here we would like to mention that the one-loop shift in the  $U(1)$  NCCS coupling depends on the regularization used. Our result (49) was obtained in the  $F^2$  regularization, in which the Yang-Mills term was added to the action and its coefficient was taken as cut-off. If we

had used dimensional regularization, the shift would be zero. This situation is not surprising, completely similar to the well-known situations for ordinary non-Abelian CS theory; see e.g. Refs. [17,20]. In our opinion, the  $F^2$  regularization is more physical, in the sense that the  $F^2$  term may naturally appear in realistic planar systems.

In field theory on ordinary spacetime, the  $U(1)$  CS coupling (CS coefficient) is known to have several interesting physical meanings, when the CS gauge field couples to various fields. For example, when there is a Maxwell term in the action,  $\kappa$  is related to the topological mass of the CS photon [14]. When there are matter fields coupled to the CS field, the CS coefficient  $\kappa$  will give rise to fractional (exchange) statistics for matter field quanta [22]. Finally a well-known folklore in the community is that an effective CS coupling for the electromagnetic field indicates the Hall effect, with the effective CS coefficient directly related to the Hall conductance. We expect all these physical interpretations should be generalizable to noncommutative spacetime. A systematic investigation of NCCS coupled to matter fields (both fermionic and bosonic) will be published elsewhere [23].

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**Notice added:** After the paper was put on the e-print archive, we are informed by e-mail from Soo-jong Rey that he has obtained the same result (in agreement with ours) on  $U(1)$  NCCS theory.



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FIG. 1. Feynman rules.

FIG. 2. One-loop Feynman diagrams in pure Chern-Simons theory (solid line-gluon;dashed line-ghost: (a)-(c) gluon self-energy, (d) ghost self-energy, (e)-(g) three-gluon vertex, (h), (i) ghost-gluon vertex.